Lecture 36  Chapts 20 & 21  Field, Potential, Multipoles, Poisson’s Equation

Announce exam distribution

Outline:

Electric Field
Electric Potential
Multipoles
Poisson Eq

Review

Coulomb’s law
\[ u(r) = \frac{1}{4\pi\varepsilon_0} \frac{q_A q_B}{Dr} \]

- \( u \) is the interaction energy (units J = V\cdot C) (note W = J/s = V\cdot C/s = V\cdot A)
- \( \varepsilon_0 \) is the permittivity of vacuum (units F/m = C/V\cdot m = C^2/J\cdot m)
- \( D = \varepsilon / \varepsilon_0 \) is the dielectric constant

Force

Of course everyone remembers that force is?
\[ \vec{f} = -\nabla u = -\frac{\partial u}{\partial r} = \frac{1}{4\pi\varepsilon_0} \frac{q_A q_B}{Dr^2} \hat{r} \]

So, forces add as vectors. This can be really a pain to keep track of. But force gives us the gateway to talk about the electrostatic field.

Field

If we have some charged particle sitting around, the field is force that particle would apply to some other particle if the second particle were present. We call the second particle the test charge – it probes the electric field. Said another way, the field represents what would happen to a particle of \( q=1 \) if it were present at any point in space.
\[ \vec{E}(r) = \frac{\vec{f}}{q_B} = \frac{1}{4\pi\varepsilon_0} \frac{q_A}{Dr^2} \hat{r} \]

(remember that force is units of N, so field is N/C or V/m)
Field is still a vector. However, an important part of the field is the field flux – the total amount of field that is perpendicular to some arbitrary surface:

\[ \Phi = \int_{\text{surface}} D \cdot d\hat{s} \]

\( s \) is the surface normal vector

If we have a single particle and we do this integral in spherical coordinates, then at some radius, \( r \), \( E \) is constant and always parallel to \( s \), so

\[ \Phi = D E \int_{\text{surface}} dS = D E (S A_{\text{sphere}}) = \frac{q_A}{4\pi e_0 r^2} \]

The distance dependence is gone because the surface area grows at exactly the same rate that the field decreases. Or, no matter how big one draws their sphere, the total flux contained is always the same – just depends on total charge.

It isn’t a big leap to realize that the surface could be any shape because the dot product ensures that ‘extra’ surface gets less impact from the non-perpendicular field. The total charge contained will be the same.

The one next step is a bit harder, but you should have all seen it before… the flux is independent of the charge’s position. We could have the charge anywhere inside of the sphere and the flux would still be \( q/\varepsilon_0 \). This also means that the total flux just depends on the total charge contained, not how it is arranged.

\[ \Phi = \sum \frac{q}{\varepsilon_0} \]

**Look familiar?** Well, it is sort of Gauss’s law (more next time)

\[ \Phi = \int_{\text{surface}} \vec{E} \cdot d\hat{s} = \int_{\text{volume}} \frac{\rho}{\varepsilon_0} dV = \sum \frac{q}{\varepsilon_0} \]

**Potential**

This leads us to our last important characterization of electrostatics, the electrostatic potential, ESP. Just as the field was a uniform way to represent the force that might affect a particle, the ESP is a way to represent the energy of a test charge. Formally, it is the work needed to move the test charge between two points. To make things consistent we always choose one test point that is at a region of zero ESP (usually \( r = \infty \)). Remember that work is just the integral of force over distance, so in this case it is the path integral of field from infinity to \( r \). (Minus sign indicates that work is done against the field rather than by the field.)
\[ \psi = \psi_r - \psi_\infty = \frac{W_{\text{test}}}{q_{\text{test}}} = -\int_\infty^r \vec{E} \cdot d\vec{l} \]

(units of ESP are work over charge, J/C)

Going back to our simple single point charge, we calculate the potential by moving radially inward from infinity to some distance \( r \). Again, \( \vec{E} \) is always parallel to our radial vector \( d\vec{r} \) so we can dispense with the dot product.

\[ \psi = -\int_\infty^r E dr' = -\int_\infty^r \frac{q_A}{4\pi\varepsilon_0 Dr'^2} dr' = -\frac{q_A}{4\pi\varepsilon_0 D} \int_\infty^r \frac{dr'}{r'^2} = -\frac{q_A}{4\pi\varepsilon_0 D} \]

remember that the interaction energy between our test particle and the charge(s) that generates the potential is just \( u(r) = q_{\text{test}}\psi(r) \)

Note that we can get the field easily from the ESP. \textbf{How might this be done?}

if force is the negative gradient of the energy, then field is the negative gradient of ESP.

\[ \vec{E} = -\nabla \psi \]

Finally, the path integral formalism above gives us a clue that just like work against gravity is conservative, work against an electric field is conservative. So, any closed path will have an electrostatic work of zero.

**Multipoles**

Last time we talked about calculating interaction energies and forces for some charges acting on other charges. One way to try and simplify the calculation is to consider groups of charges as single entities. Thus, instead of talking about the field from two charges, we could talk about the field from one dipole. Four charges are a quadrupole, eight an octopole and so on.

Remember from last time that for a single charge \( \psi = \frac{1}{4\pi\varepsilon_0} \frac{q_A}{r^2} \)

It turns out that for a distribution of charge (say a blob with charge density, \( \rho \))

\[ \psi = \frac{1}{4\pi\varepsilon_0} \int_{\text{Volume}} \frac{\rho}{r^2} dV \]

If you try to solve this it turns out that you will derive the Legendre polynomials such that

\[ \psi = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_{\text{Volume}} (r')^n P_n(\cos \theta) \rho dV \]
(sketch diagram, r is dist. from center of volume to test point, r’ is dist from center of volume to volume element, r” is distance from volume element to test point. Theta is angle between r and r’.)

OR

\[
\psi = \frac{1}{4\pi \varepsilon_0} \left[ \frac{1}{r} \int_{\text{Volume}} \rho dV + \frac{1}{r^2} \int_{\text{Volume}} r' \cos \theta \rho dV + \frac{1}{r^3} \int_{\text{Volume}} (r')^2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \rho dV + \cdots \right]
\]

(Notice the subtle difference in r. Before we had to integrate over r”, which was a nasty variable to handle. Now, we are integrating over r’ which is only within our volume. So it is nasty, but we only have to deal with it one time and then as long as our object (volume) doesn’t change we don’t have to re-evaluate the integral. The thing that is now difficult to handle is r but that is a constant, from some fixed point – the center – of the volume. Does everyone see how this has made our lives easier? The \( \rho dV \) integral is now pretty easy.)

So, we have made our favorite thing – a power series, an expansion!

Normally we rewrite things a bit:

\[
\psi = \psi_{\text{mon}} + \psi_{\text{dip}} + \psi_{\text{quad}} + \cdots
\]

where

\[
\psi_{\text{mon}} = \frac{q_{\text{tot}}}{4\pi \varepsilon_0 r}
\]

\[
\psi_{\text{dip}} = \frac{\hat{r} \cdot \hat{\mu}}{4\pi \varepsilon_0 r^2}
\]

where \( \hat{\mu} = \int_{\text{Volume}} \hat{r}' \rho dV = \sum_{i=1}^{n} q_i \hat{r}_i \)

and so on with more complicated tensor stuff as we continue.

Remind me how this has made our lives easier? We can calculate the monopole, dipole, quad etc. moments ahead of time – those do not have anything to do with anything except the charge distribution. Then it is relatively easy for us to calculate the potential or the interaction energy with some other charge (or dipole or…) later on.

When is the dipole term really important? when the total charge is zero.

For example, let’s say we have two particles of charge +q and -q some distance, a apart. (like an HCl molecule) The dipole moment is (with the origin at the center):
\[
\vec{\mu} = q \frac{a}{2} \hat{x} + (-q) \frac{a}{2} (-\hat{x}) = qa \hat{x}
\]

(note that we could have chosen any origin)

The electrostatic potential is:

\[
\psi_{\text{dip}} = \frac{\vec{\hat{r}} \cdot \vec{\mu}}{4\pi D\epsilon_0 r^2} = \frac{qa \hat{x} \cdot \vec{\hat{r}}}{4\pi D\epsilon_0 r^2} = \frac{qa \cos \theta}{4\pi D\epsilon_0 r^2}
\]

(this is a lot easier than if we had done the individual charges  Ex. 21.5)

The interaction energy with a particle of charge +q is:

\[
u = q \psi_{\text{dip}} = \frac{qq_a \cos \theta}{4\pi D\epsilon_0 r^2}
\]

Note that
- charge-charge interaction \( r^{-1} \)
- charge-dipole interaction \( r^{-2} \)
- dipole-dipole interaction \( r^{-3} \)

Poisson Equation

What if we have a ridiculously complicated set of charges. Well, we could keep many terms in our multipole expansion. Or, we could do something a bit more clever.

Remember Gauss’s Law that we had last time (I think I left out the \( D \)):

\[
\int_{\text{surface}} \int_{\text{volume}} \vec{D} \cdot \vec{E} \cdot d\vec{s} = \int_{\text{volume}} \rho \frac{dV}{\epsilon_0}
\]

If we compare this to Gauss’s more general Theory:

\[
\int_{\text{surface}} \int_{\text{volume}} \vec{D} \cdot d\vec{s} = \int_{\text{volume}} D \vec{\nabla} \cdot \vec{E} dV
\]

We see that:

\[
D \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{or} \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{D \epsilon_0}
\]

You may recognize this as one of Maxwell’s equations (quite different than Maxwell’s relations) though you may never have seen the differential form before. We can also restate this equation in terms of the potential which is often more useful:

\[
\vec{E} = -\nabla \psi
\]

so

\[
\nabla^2 \psi = -\frac{\rho}{D \epsilon_0}
\]
This is Poisson’s equation. Big stuff. It gives us an easy way to solve for the potential even if we have a really complicated set of charges that sit within different media with varying dielectric constants. Note that if we have just point charges surrounded by a uniform dielectric, then Poisson’s equation reduces to Coulomb’s law.

Please take a good look at the books examples, such as 21.6